

An ergodic diffusion with unbounded inward and outward drift

Horst Thaler

Department of Mathematics and Informatics, University of Camerino,
Via Madonna delle Carceri 9, I-62032, Camerino (MC), Italy;
E-mail: horst.thaler@unicam.it

Abstract

It is argued that a diffusion may be ergodic even though the drift field has unbounded outward-directed parts. The discussion employs stochastic and numerical methods.

Keywords: stochastic differential equation, diffusion process, ergodicity, transience, recurrence.

Mathematics Subject Classification (2010). 37A50, 60H15, 60J60.

1 Introduction

We shall investigate the ergodicity of the following stochastic differential equation (SDE)

$$\begin{cases} dX_t = b(X_t)dt + dW_t, \\ X_0 = \zeta, \end{cases} \quad (1)$$

where $\zeta \in \mathbb{R}^2$ and $(W_t)_{t \geq 0}$ is a 2-dimensional Brownian motion. The drift $b : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by

$$b_1 = -4x_1^3 + 12x_1x_2^2, \quad b_2 = -12x_1^2x_2 + 4x_2^3. \quad (2)$$

Here a stochastic process $X = (X_t)_{t \geq 0}$ is termed ergodic if there exists a

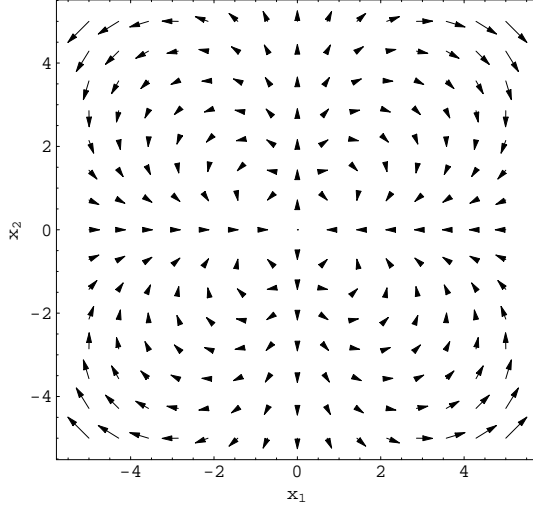


Figure 1: The drift field b according to (2)

unique invariant probability measure μ for X and if for every $f \in L^1(\mathbb{R}^d, \mu)$ it holds that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(X_t) dt = \int_{\mathbb{R}^d} f d\mu, \quad P_\zeta - \text{a.s.}, \text{ for all } \zeta \in \mathbb{R}^d, \quad (3)$$

where P_ζ denotes the law of X with starting point ζ . Condition (3) is also called the law of large numbers of the process X . Under very general assumptions it can be shown that the existence of an invariant probability measure is equivalent to X being positive recurrent, see [6, Theorem 4.1]. As we shall see it is difficult to classify the diffusion X , which solves equation (1), as being transient, recurrent or positive recurrent. This is due to the funny drift b itself. Before proceeding we fix some notation.

Notation: $\mathbb{E}_\zeta[\cdot]$ shall denote the expectation with respect to the law P_ζ . The standard inner product of two vectors $x, x' \in \mathbb{R}^d$ is written as $x \cdot x'$. We set $r = r(x) := \sqrt{x \cdot x}$. Furthermore, $B(\mathbb{R}^d)$ is the space of bounded, real-valued measurable functions and $C^2(\mathbb{R}^d)$ stands for the space of twice continuously differentiable functions. The indicator function of a set A will be denoted by 1_A .

Let us have a closer look at b defined in (2). Let $e_r := (x_1, x_2)/r$ be the

radial unit vector field, then we may distinguish two cases

$$\begin{aligned} (a) \quad & e_r \cdot b < 0, \text{ if } |x_2| < |x_1|, \\ (b) \quad & e_r \cdot b > 0, \text{ if } |x_2| > |x_1|. \end{aligned} \tag{4}$$

Therefore, in case (a) we have an inward drift and in case (b) an outward drift. Moreover, evaluating $e_r \cdot b$ with $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$, $\varphi \in [-\pi, \pi]$, we see that in both cases there results a growth behavior proportional to r^3 and that the proportionality constant is positive and increases as $|\varphi| \rightarrow \pi/2$ for $\{\pi/4 < |\varphi| < 3\pi/4\}$ and decreases, being negative, as $|\varphi| \rightarrow 0$ or $|\varphi| \rightarrow \pi$ for $\{|\varphi| < \pi/4\}$ and $\{|\varphi| > 3\pi/4\}$, respectively. It equals 0 for $|\varphi| = \pi/2$, $|\varphi| = 3\pi/4$. This means that moving away from the origin there are areas with unbounded inward drift as well as areas with unbounded outward drift. Moreover, it should be noted that $\text{curl}(b_1, b_1) = -12x_1x_2$ which shows that it is impossible to write the drift as $b = -\nabla V$ for some differentiable function V .

The paper is organized as follows. In section 2 we collect some background material on the different large time asymptotics that a stochastic process may show. These properties are discussed in terms of the coefficients of the generator of X . In section 3 we return to the investigation of the diffusion solving (1). It is made evident that the latter diffusion cannot be handled using the standard criteria of section 2. Finally, the numerical results for the long-term behavior of (3) are presented which give strong indications that the diffusion solving (1) is ergodic.

2 Criteria for transience and recurrence

Here we discuss criteria that allow for a complete classification of a diffusion, at least if the latter are applicable. To this end we consider a class of more general SDEs, namely

$$\begin{cases} dX_t = b(X_t)dt + dW_t, \\ X_0 = \zeta, \end{cases} \tag{5}$$

where $(W_t)_{t \geq 0}$ is a d -dimensional Brownian motion and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a smooth vector field. The semigroup $(S_t)_{t \geq 0}$, $S_t : B(\mathbb{R}^d) \rightarrow B(\mathbb{R}^d)$, associated with X is given by $S_t f(\zeta) := \mathbb{E}_\zeta [f(X_t) 1_{\{t < \eta\}}]$, where η denotes the explosion

time of X . We shall assume that the generator L of X uniquely determines $(S_t)_{t \geq 0}$ and that on $C^2(\mathbb{R}^d) \subset D(L)$ it is given by

$$L = \frac{1}{2}\Delta + b\nabla, \quad \text{with formal adjoint } L^*(\cdot) = \operatorname{div}((\frac{1}{2}\nabla - b)(\cdot)).$$

Since a priori we do not demand the semigroup to be strongly continuous the generator L might be given also in the weak sense, see e.g. [4, Section 2.2] and [2, Section 2.3]. From now on processes sharing the foregoing properties will simply be called diffusions.

In [1] Bhattacharya has given quite general sufficient criteria in terms of the coefficients b_i that allow to decide whether a diffusion is transient, recurrent or that it even admits a finite invariant measure. We recall that the diffusion X is said to be recurrent if for any $\zeta \in \mathbb{R}^d$ and any open set $U \subset \mathbb{R}^d$ we have

$$P_\zeta(X_{t_n} \in U \text{ for a sequence of } t_n' \text{'s increasing to } \infty) = 1.$$

If for any $\zeta \in \mathbb{R}^d$ it holds that

$$P_\zeta(\lim_{t \rightarrow \infty} r(X_t) = \infty) = 1,$$

then X is called a transient process. Finally, X is said to be positive recurrent, if $\mathbb{E}_\zeta[\sigma_U] < \infty$, for all $\zeta \in \mathbb{R}^d$ and all open sets $U \subset \mathbb{R}^d$, where σ_U is the first entrance time of U .

Adopting the same notation as in [1], we introduce the following functions

$$\begin{aligned} C(x) &= 2 \sum_{i=1}^d x_i b_i(x) \\ \bar{\beta}(r) &= \sup_{|x|=r} (d-1 + C(x)), \quad \underline{\beta}(r) = \inf_{|x|=r} (d-1 + C(x)) \\ \bar{I}(r) &= \int_{r_0}^r \frac{\bar{\beta}(u)}{u} du, \quad \underline{I}(r) = \int_{r_0}^r \frac{\underline{\beta}(u)}{u} du \end{aligned} \tag{6}$$

With these preparations we may quote the following two theorems from [1]

Theorem 2.1 (a) *If for some $r_0 > 0$*

$$\int_{r_0}^{\infty} \exp\{-\bar{I}(u)\} du = \infty, \tag{7}$$

then the diffusion with generator L is recurrent.

(b) If for some $r_0 > 0$

$$\int_{r_0}^{\infty} \exp\{-\underline{I}(u)\} du < \infty, \quad (8)$$

then the diffusion with generator L is transient.

Theorem 2.2 (a) The diffusion with generator L is recurrent and admits a finite invariant measure (unique up to a constant multiple) if there exists a $r_0 > 0$ such that (7) holds and

$$\int_{r_0}^{\infty} \exp\{\bar{I}(u)\} du < \infty. \quad (9)$$

(b) If there exists a $r_0 > 0$ such that (7) holds and

$$\lim_{N \rightarrow \infty} \int_{r_0}^N \exp\{-\bar{I}(s)\} \left(\int_{r_0}^s \exp\{\bar{I}(u)\} du \right) ds \Big/ \int_{r_0}^N \exp\{-\underline{I}(u)\} du = \infty, \quad (10)$$

then the recurrent diffusion does not admit a finite invariant measure.

Henceforth a unique finite invariant measure will always be normalized to become a probability measure. In the following examples we explore some possible scenarios. For simplicity only radially symmetric problems are considered and we set $I(r) := \bar{I}(r) = \underline{I}(r)$.

Example 2.1 (a) Let $b = 0$ in (5), then $X_t = W_t$ and an elementary calculation gives by (7) and (8) that Brownian motion is recurrent in $d = 1, 2$ and transient for $d \geq 3$.

(b) Let $V(x) = -r^{-\alpha}$ and $b = -\nabla V$, with $\alpha > 0$. In this case we find

$$I(r) = \int_{r_0}^r \frac{(d-1-2\alpha u^{-\alpha})}{u} du = (d-1)(\ln r - \ln r_0) + 2(r^{-\alpha} - r_0^{-\alpha}).$$

By elementary calculations one deduces that (7) holds for $d = 1, 2$, implying recurrence, and that (8) is valid for $d \geq 3$ entailing transience. On the other hand, evaluating the quotient of the l.h.s in (10) gives the following estimate

$$\text{quotient of integrals of (10)} \geq C(r_0)N,$$

for N large enough, with $C(r_0)$ some positive constant. This shows that there is no finite invariant measure for any dimension. Still there is a σ -finite invariant measure $\nu(dx) = e^{-2V}dx$ for every dimension, since the stationary Fokker-Planck equation has a solution

$$L^*(e^{-2V}) = \operatorname{div}((\tfrac{1}{2}\nabla - b)e^{-2V}) = 0.$$

(c) Let $V(x) = r^\alpha$ and $b = -\nabla V$, where $\alpha > 0$. Here we obtain

$$I(r) = \int_{r_0}^r \left(\frac{(d-1)}{u} - 2\alpha u^{\alpha-1} \right) du = \ln r^{d-1} - 2r^\alpha - \ln r_0^{d-1} + 2r_0^\alpha,$$

which entails properties (7) and (9). Now the diffusion is recurrent with finite invariant measure.

Remark. The functions V in examples (b) and (c) are not smooth at the origin. But the latter may be easily rendered such.

It is clear that the diffusive part dW_t tends to spread a “particle” moving according to (5). This spreading may be compensated by a sufficiently strong drift b . As can be seen from the previous example (c), if the driving field b is inward directed and the gradient of a potential that grows like a positive power, then we are guaranteed an invariant probability measure μ on \mathbb{R}^d . In fact in the case of example (c) the diffusion is even ergodic, meaning that for every $f \in L^1(\mathbb{R}^d, \mu)$ we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(X_t) dt = \int_{\mathbb{R}^d} f d\mu, \quad P_\zeta - \text{a.s.}, \text{ for all } \zeta \in \mathbb{R}^d. \quad (11)$$

Notice that, for f bounded and measurable, property (11) immediately implies

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S_t f(\zeta) dt = \int_{\mathbb{R}^d} f d\mu, \quad \zeta \in \mathbb{R}^d. \quad (12)$$

A large class of processes for which properties (11) and (12) hold is given by recurrent diffusions with invariant probability measure that satisfy in addition the following conditions (cf. [8, Section 1.3])

Condition (A) The semigroup $(S_t)_{t \geq 0}$ maps $C_0(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$ and is strongly continuous.

Condition (B) There is a Borel measure ν on \mathbb{R}^d with support on \mathbb{R}^d and a strictly positive function $p_t(x, y)$ continuous for $(t, x, y) \in (0, \infty) \times$

$\mathbb{R}^d \times \mathbb{R}^d$ such that the transition probability can be written as $K_t(x, dy) = p_t(x, y)\nu(dy)$.

3 An ergodic diffusion with strong outward drift

We now come to the discussion of the diffusion $(X_t)_{t \geq 0}$ which solves (1). The peculiar form of (2) is related to a complex differentiation, namely if $z = x_1 + ix_2$ and $g = (x_1 + ix_2)^4$, then

$$b_1 = -\operatorname{Re} \left(\frac{d}{dz}(g) \right), \quad b_2 = -\operatorname{Im} \left(\frac{d}{dz}(g) \right).$$

We know at least that X has continuous sample paths, maps $B(\mathbb{R}^d)$ into $C(\mathbb{R}^d)$ and is conservative, i.e. its explosion time is almost surely infinite. The first property is a consequence of our assumptions, see e.g. [5]. The second and third properties are consequences of Lemma 2.5 and condition (4.1) of [1].

Let us turn to the transience and recurrence properties of our diffusion. The absolute maxima and minima of $(1 + C(x))$ given the constraint $|x| = r$ are found to be $\bar{\beta}(r) = 1 + 8r^4$ and $\underline{\beta}(r) = 1 - 8r^4$, respectively. This gives $\bar{I}(r) = \ln r - \ln r_0 + 2(r^4 - r_0^4)$ and $\underline{I}(r) = \ln r - \ln r_0 - 2(r^4 - r_0^4)$, hence

$$\int_{r_0}^{\infty} \exp\{-\bar{I}(u)\} du = r_0 e^{2r_0^4} \int_{r_0}^{\infty} \frac{1}{u} e^{-2u^4} du < \infty \quad (13)$$

$$\int_{r_0}^{\infty} \exp\{-\underline{I}(u)\} du = r_0 e^{-2r_0^4} \int_{r_0}^{\infty} \frac{1}{u} e^{2u^4} du = \infty. \quad (14)$$

This shows that none of the criteria (7) and (8) applies. As far as the existence of a finite invariant measure is concerned we use the estimate

$$\text{quotient of integrals of (10)} \leq C(r_0) \int_{r_0}^N (s - r_0) ds \Big/ \int_{r_0}^N \frac{1}{u} e^{2u^4} du, \quad (15)$$

for some positive constant $C(r_0)$. Since the r.h.s of (15) tends to 0 as $N \rightarrow \infty$ criterion (10) is not valid. Hence, even if the diffusion were recurrent we could not tell whether it possesses an invariant probability measure.

In order to get some insight into the problem of the diffusion (1) several numerical tests have been performed whose aim was to study the behavior of

$$f_T := \frac{1}{T} \int_0^T f(X_t) dt, \quad (16)$$

as T increases. As f_T is an expression depending on the sample paths a strong Taylor approximation (of order 1.5) was used for the latter. More precisely, the strong Taylor scheme is implemented through (cf. [7, p. 355, eq. 4.12])

$$Y_{n+1}^k = Y_n^k + b_k \Delta + \Delta W^k + \frac{1}{2} L^0 b_k \Delta^2 + L^k b_k \Delta Z^k, \quad k = 1, 2, \quad (17)$$

where

$$\Delta W^k = U_1^k \sqrt{\Delta}, \quad \Delta Z^k = \frac{1}{2} \Delta^{3/2} (U_1^k + \frac{1}{\sqrt{3}} U_2^k),$$

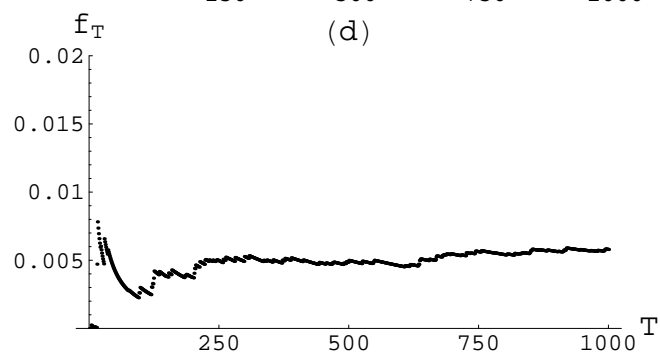
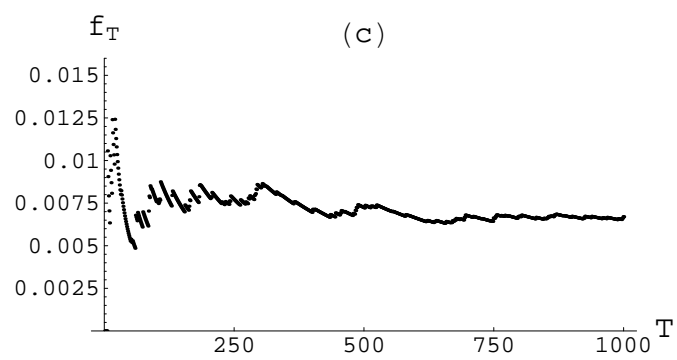
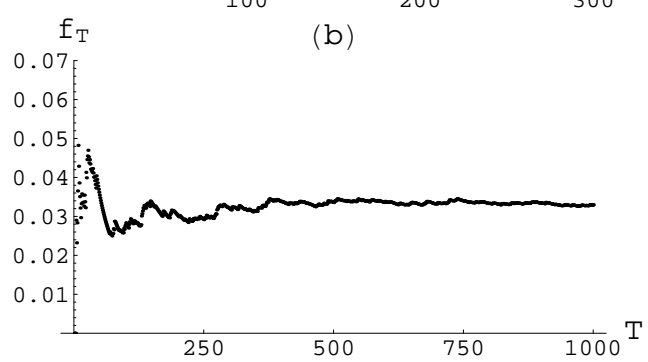
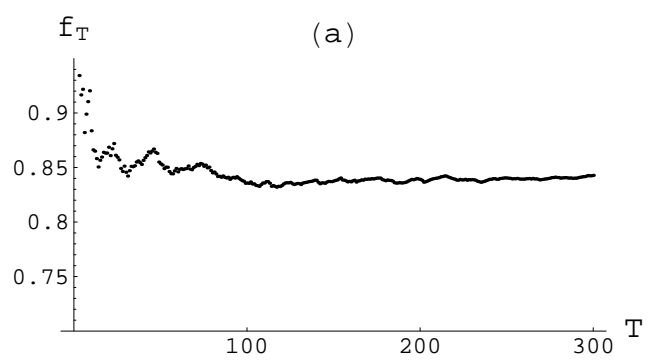
with U_1^k, U_2^k being independent standard Gaussian random variables and where Δ is the time step. Moreover,

$$L^0 = b_1 \partial_1 + b_2 \partial_2 + \frac{1}{2} (\partial_1^2 + \partial_2^2), \quad L^1 = \partial_1, \quad L_2 = \partial_2.$$

The corresponding results are shown in Figure 2 with $\Delta = 10^{-4}$. Note that for $f = 1_A$, A a Borel set, equality (3) entails

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T 1_A(X_t) dt = \mu(A). \quad (18)$$

Hence indicator functions comprise a natural class of test functions to be used. A glance at Figure 2 clearly displays a convergent behavior of expression (16) for different starting points and different positions of the indicator functions. One also sees that the limits decay rather fast and that the diffusion takes ever more time to stabilize the farther away we move from the origin. On the other hand, if convergence of (18) is observed then the limit μ is a natural candidate for an invariant probability measure. Note that according to [2, Theorem 2.2.5] all transition probabilities $K_t(x, dy)$ of the diffusion X are equivalent which by Doob's theorem, see [3, Theorem 4.2.1], gives the uniqueness of the invariant measure μ . This means that the numerical findings support the assertion on ergodicity of X .



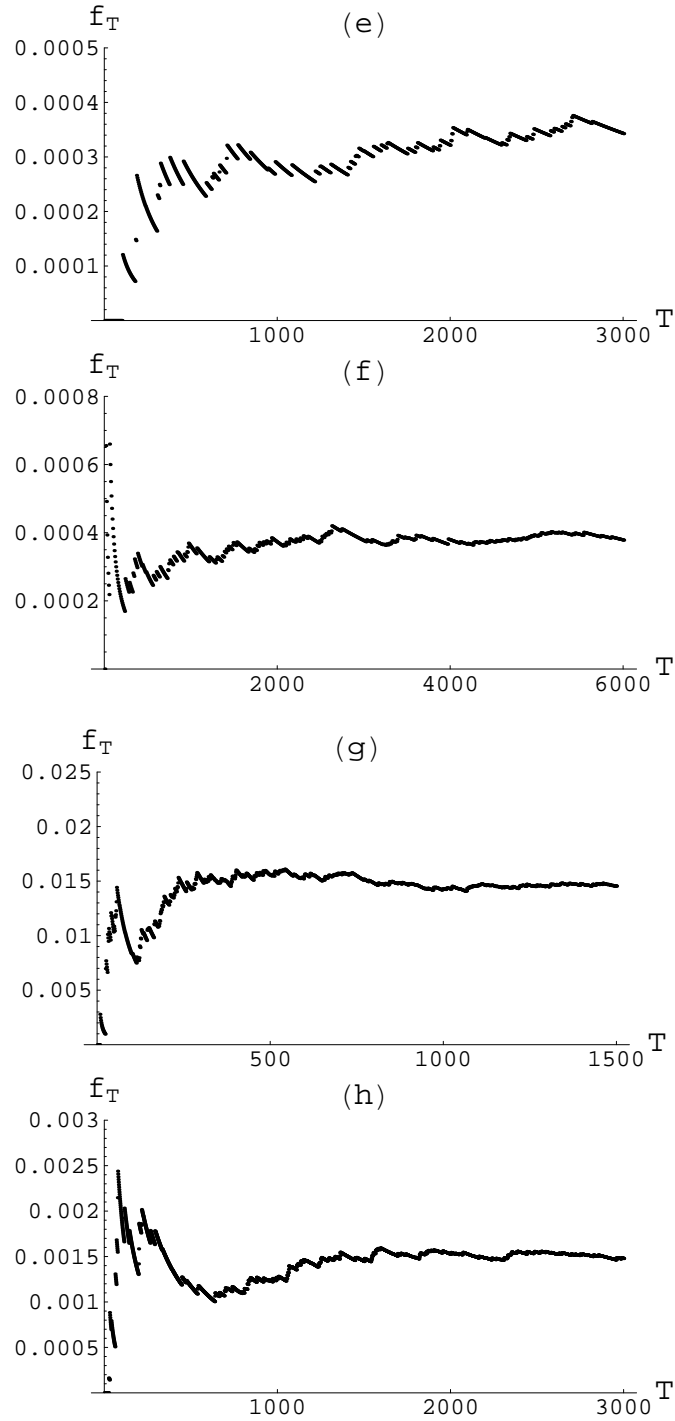


Figure 2: Behavior of f_T , with $f = 1_{B_1}(x - x_0)$ where $B_1 = \{x \in \mathbb{R}^2 \mid r(x) < 1\}$ and for x_0 the following values have been chosen. (a) $(0, 0)$, (b) $(2, 0)$, (c) $(0, 2)$, (d) $(3, 0)$, (e) and (f) $(0, 3)$, (g) $2(1/\sqrt{2}, 1/\sqrt{2})$, (h) $3(1/\sqrt{2}, 1/\sqrt{2})$. The starting points ζ have been picked randomly from $[-10, 10]^2$.

4 Acknowledgments

The author would like to thank Simona Bernabei for her careful reading of the manuscript. He also wishes to express his gratitude for the financial support through the program “Rientro dei cervelli” of the italian M.I.U.R.

References

- [1] R. N. Bhattacharya, *Criteria for recurrence and existence of invariant measures for multidimensional diffusions*, Ann. Probab. **6**, no. 4, (1978) 541-553.
- [2] M. Bertoldi and L. Lorenzi, *Analytical Methods for Markov Semigroups*, Pure and Applied Mathematics (Boca Raton), 283, Chapman and Hall/CRC, 2007.
- [3] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, New-York, 1996.
- [4] E.B. Dynkin, *Markov Processes*, Vol. 1, Springer-Verlag, Berlin, 1965.
- [5] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland Publishing Company, Amsterdam 1981.
- [6] W. Kliemann, *Recurrence and invariant measures for degenerate diffusions*, Ann. Probab. **15**, no. 2, (1987), 690-707.
- [7] P.E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 1992.
- [8] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, Cambridge Studies in Advanced Mathematics, 24, Cambridge University Press, Cambridge 1990.